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DERIVATIONS WITH ONLY DIVISORIAL SINGULARITIES ON RATIONAL AND RULED SURFACES

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Introduction

Let k be an algebraically closed field of characteristic $p>0$. In 1979, Piotr Blass suggested the folowing problem (in case X is the projective plane):

(*) Given a nonsingular surface X, find all inseparable coverings $Y\to X$ of degree p, with Y nonsingular.

Blass's problem was solved recently by Spencer Bloch; he showed that $\mathbb{P}^2$ has no such nonsingular coverings.

In this paper we solve (*) for the case $X=\mathbb{F}_n$, where the $\mathbb{F}_n$ are Nagata's rational surfaces: It turns out that $\mathbb{F}_n$ has exactly one such covering if $p\nmid n$, and exactly two if $p\mid n$ (Theorem 2.2). We also give an elementary proof of Bloch's result (Theorem 2.1). It is not clear to us how to treat more general rational surfaces: We give examples (2.3 and the discussion preceding it) which indicate that solutions of (*) do not translate easily across even a single blowing-up or blowing-down.

The key to the solution of (*) for $\mathbb{P}^2$ and $\mathbb{F}_n$ is the fact, due to Rudakov and Šafarevič ([6], Theorem 4), that an inseparable degree p covering $Z\to Y$ of nonsingular surfaces is the quotient morphism with respect to a p-closed derivation D on Z without isolated singularities. We reprove this result (Theorem 1.2), using an elementary fact (see [1]) about power series rings in two variables sandwiched by the Frobenius. Given the Rudakov–Šafarevič result, (*) is the same as

(**) Let Z be the inverse Frobenius of X. Find all p-closed derivations of $k(Z)$ without isolated singularities on Z.

In the cases mentioned above this involves only high school mathematics, modulo some knowledge of the intersection theory on Z.

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In the language of vector bundles, (**) may be rephrased as follows:

(***) Find all integrable sublinebundles (in the sense of [7], §3) of the tangent bundle T_Z of Z.

Our results on $\mathbb{F}_n$ translate as follows: Let

$$0 \to \Phi \to T_{\mathbb{F}_n} \to N \to 0 \tag{†}$$

be the canonical exact sequence of vector bundles, where N is the pullback of the tangent bundle on $\mathbb{P}^1$ and $\Phi = T_{\mathbb{F}_n/\mathbb{P}^1}$ is the bundle of tangents along the fibres of $\mathbb{F}_n$. Then (†) splits if and only if $p \mid n$. The integrable sublinebundles of $T_{\mathbb{F}_n}$ are Φ and, if $p \mid n$, those given by integrable sections $N \to T_{\mathbb{F}_n}$. All of these latter are derived from one such by automorphisms of $\mathbb{F}_n$.

Finding all solutions to (**) looks like a rather complicated task in case Z is a ruled surface over a nonrational curve C. We make some tentative efforts in this direction by studying the case $Z = C \times \mathbb{P}^1$, mainly to elucidate the observation that a point of order p on the Jacobian of C gives rise to a nontrivial ruled surface Y over the Frobenius of C whose pullback to C is $C \times \mathbb{P}^1$, and to describe a 'supersingular' analogue of this phenomenon.

1. Generalities

Given a variety Z over k, we denote by $\theta : Z \to \theta Z$ the Frobenius of Z.

1.1. Lemma. *Let X be a k-variety, and let $Z = \theta^{-1}X$ be its inverse Frobenius. Suppose that X (equivalently, Z) is normal, and that Y is a normal k-variety. The following are in natural* 1–1 *correspondence:*

(1) *purely inseparable coverings $Y \xrightarrow{f} X$ of exponent* 1,
(2) *purely inseparable coverings $Z \xrightarrow{\pi} Y$ of exponent* 1,
(3) *factorizations $Z \xrightarrow{\pi} Y \xrightarrow{f} X$ of the Frobenius of Z, with f, π as in* (1), (2).

Proof. Given (1), we have $K = k(X) \subset k(Y) \subset K^{1/p}$. Let $\operatorname{Spec} A$ be an affine open subset of X, and $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$. We have $A \subset B$ and $B^p \subset K$. Since B is integral over A and A is normal, $B^p \subset A$. Hence $B \subset A^{1/p}$, and we obtain π as in (3). Given (2), we have a factorization $\theta^{-1}Y \xrightarrow{f'} Z \xrightarrow{\pi} Y$, giving $Y \xrightarrow{f} X$, where $f = \theta f'$ is as in (3).

Let Z and Y be normal k-varieties. Let $\pi : Z \to Y$ be an inseparable covering of degree p. We recall some facts about such morphisms from [6]. By Jacobson's theory ([3], Ch. IV §8) there exists a derivation D of $k(Z)$ such that

$$k(Y) = k(Z)^D := \{f \in k(Z) \mid D(f) = 0\}. \tag{1}$$

D is *p-closed* (i.e. $D^p = aD$ with $a \in k(Z)$), and determined by π up to a factor in $k(Z)^*$. (If $D' = gD$ with $g \in k(Z)^*$, we will call D and D' *equivalent*, and write $D \sim D'$.) Let $q \in Z$. Since Y is normal, we have

$$\mathcal{O}_{Y,\pi(q)} = k(Z)^D \cap \mathcal{O}_{Z,q}. \tag{2}$$

Conversely, given a nonzero p-closed derivation D of $k(Z)$, we let π = identity map (set-theoretically) and use (2) to define a normal algebraic variety Y on the topological space underlying Z. Then $\pi : Z \to Y$ is an inseparable covering of degree p.

Now assume further that Y and Z are surfaces, with Z nonsingular. If $\xi, \eta \in \vartheta := \mathcal{O}_{Z,q}$ are regular parameters at q, we can write

$$D = b_q\left(f\frac{\partial}{\partial\xi} + g\frac{\partial}{\partial\eta}\right), \tag{3}$$

where $f, g \in \vartheta$ and

$$I(D,q) = (f,g)\vartheta \tag{4}$$

is an ideal of height ≥ 2. By the chain rule, $I(D,q)$ is independent of the choice of parameters ξ, η and b_q is determined up to a unit in ϑ. Note that $D_q := b_q^{-1}D \sim D$, $D_q(\vartheta) \subset \vartheta$ and $\mathcal{O}_{Y,\pi(q)} = \vartheta^{D_q}$.

The divisor on Z with local equation b_q at q will be called the *divisor* (D) *of* D. We say that D has *only divisorial singularities* (or is *without isolated singularities*) *on* Z if $I(D,q) = \mathcal{O}_{Z,q}$ for all $q \in Z$.

1.2. Theorem. ([6], Theorems 1 and 4.) *Let $\pi : Z \to Y$ be an inseparable covering of degree p, where Z and Y are algebraic surfaces with Z nonsingular and Y normal. Then π is the quotient morphism with respect to a nonzero p-closed derivation D on Z. Y is nonsingular if and only if D has no isolated singularity on Z.*

Proof. The first part of the theorem has already been established. As for the second statement, the sufficiency follows from [7], Proposition 6, as noted in [6], Theorem 1. (The restriction to *complete* local rings in [6] is unnecessary.) For the necessity, consider $q \in Z$, and let $\vartheta = \mathcal{O}_{Z,q}$, $\vartheta' = \mathcal{O}_{Y,\pi(q)}$. The derivation D_q of ϑ extends uniquely to a derivation, which we also call D_q, of the completion $\hat{\vartheta}$, and $\hat{\vartheta}' = \hat{\vartheta}^{D_q}$. We recall the following fact ([1], Theorem):

If R, A are power series rings in two variables over k, with $R^p \subsetneq A \subsetneq R$, then there exist $x, y \in R$ such that $R = k[[x,y]]$ and $A = k[[x, y^p]]$. We have $fx_\xi + gx_\eta = D_q(x) = 0$. Since $\partial(x,y)/\partial(\xi,\eta)$ is a unit in $\hat{\vartheta}$, we have that $f \mid g$ or $g \mid f$ in $\hat{\vartheta}$. This immediately gives a prime divisor of $f, g \in \vartheta$ (contradicting the choice of f, g), unless f or g is a unit in ϑ, which means that D has no isolated singularity at q. (See also [5], Lemma 1.3.)

1.3. Corollary. *Inseparable degree p coverings $Y \to X$ of nonsingular surfaces are* *in natural* 1–1 *correspondence with quotient morphisms* $\theta^{-1}X = Z \to Y$ *with respect to nonzero p-closed derivations of* $k(Z)$ *without isolated singularities on* Z.

Proof. By 1.1 and 1.2.

2. Applications to rational surfaces

2.1. Theorem (Bloch). *There is no inseparable degree p covering* $Y \to \mathbb{P}^2$ *with* Y *nonsingular.*

Proof. We show that every nonzero derivation on $\mathbb{P}^2$ has an isolated singularity; by 1.3 this is more than enough. We regard $\mathbb{P}^2$ as the union of affine planes with coordinates (x, y), $(t = 1/x,\ u = y/x)$, and $(v = 1/y,\ w = x/y)$. We denote by L the line at infinity in the (x, y)-coordinate system (given by $t = 0$ or $v = 0$) and by P, Q the points $t = u = 0$, $v = w = 0$, respectively.

Given a derivation D of $k(x, y)$, we may replace D by an equivalent derivation of the form $D = f(\partial/\partial x) + g(\partial/\partial y)$, where $f, g \in k[x, y]$ are coprime. If D has no isolated singularity on $\mathbb{P}^2$, then f and g have no common zero.

Put $r = \deg f$, $s = \deg g$. We leave as an exercise the fact that, if $fg = 0$ or $r \neq s$, then D has an isolated singularity at P or Q. We assume that f and g are nonzero and $r = s$. Define $\bar{f}, \bar{g} \in k[t, u]$, $\tilde{f}, \tilde{g} \in k[v, w]$ by

$$f = \bar{f}x^r = \tilde{f}y^r, \qquad g = \bar{g}x^r = \tilde{g}y^r.$$

Then $\bar{f}, \bar{g}$ are coprime and not divisible by t, and $\tilde{f}, \tilde{g}$ are coprime and not divisible by v. We have

$$D = \bar{f}t^{-r}\left(-t^2 \frac{\partial}{\partial t} - tu \frac{\partial}{\partial u}\right) + \bar{g}t^{-r}\left(t \frac{\partial}{\partial u}\right) \sim (\bar{g} - \bar{f}u) \frac{\partial}{\partial u} - \bar{f}t \frac{\partial}{\partial t};$$

similarly

$$D \sim (\tilde{f} - \tilde{g}w) \frac{\partial}{\partial w} - \tilde{g}v \frac{\partial}{\partial v}.$$

Put $h = xg - yf$. h is nonzero – otherwise f and g meet at $x = y = 0$. We have $\deg h \leq r \Leftrightarrow t \mid \bar{g} - \bar{f}u \Leftrightarrow v \mid \tilde{f} - \tilde{g}w$. Suppose this is not the case. Then, if $r = 0$, D has an isolated singularity at $t = 0$, $u = g/f$. If $r > 0$, the singularity occurs at a common point of f and g on L; there are such by Bezout's theorem. We assume therefore that $\deg h \leq r$. Putting $h = \bar{h}x^r = \tilde{h}y^r$, with $\bar{h} \in k[t, u]$, $\tilde{h} \in k[v, w]$, we have

$$D \sim \bar{h} \frac{\partial}{\partial u} - \bar{f} \frac{\partial}{\partial t} \sim \tilde{h} \frac{\partial}{\partial w} + \tilde{g} \frac{\partial}{\partial v}.$$

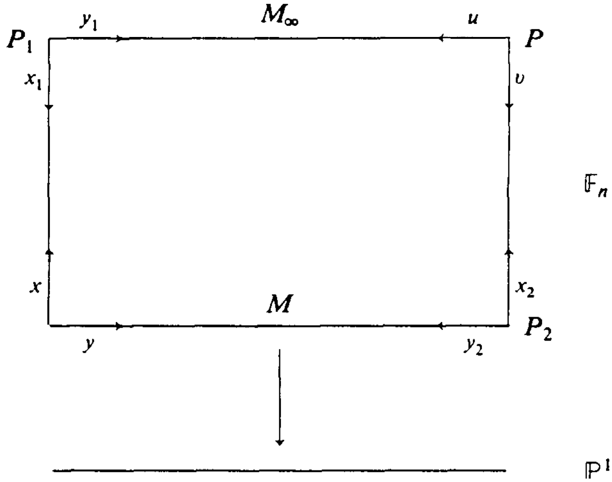
Clearly $\bar{f}, \bar{h}$ and $\tilde{g}, \tilde{h}$ are coprime. Also $\deg h < r \Leftrightarrow t \mid \bar{h} \Leftrightarrow v \mid \tilde{h}$. Since f and g meet on L by Bezout, $\tilde{g}$ meets v or $\bar{f}$ meets t. Hence we may assume that $\deg h = r$, $\bar{f}$ and $\bar{h}$ do not meet on t, and $\tilde{g}$ and $\tilde{h}$ do not meet on v.

We claim that $\tilde{h}(Q)=0$; for this we may assume $x \nmid h$. Then

$$f\cdot h := \dim_k k[x,y]/(f,h) = f\cdot x \leq r-1$$

since $x \nmid f$ and x divides the degree form of f. So $(f\cdot h)_L = (\tilde{f}\cdot\tilde{h})_Q \geq r^2-(r-1)>0$, and $\tilde{h}(Q)=0$. So $\tilde{g}(Q)\neq 0$. Then $r^2=(f\cdot g)_L=\tilde{f}\cdot\tilde{g}=\tilde{f}\cdot(\tilde{g}-\tilde{f}u)=\tilde{f}\cdot t\tilde{h}=\tilde{f}\cdot t= f\cdot L-(f\cdot L)_Q \leq r-1$, and absurdity, which completes the proof of 2.1.

We now turn to finding all inseparable degree p coverings of $\mathbb{F}_n$ by nonsingular surfaces.



We regard $\mathbb{F}_n$ as the patching of affine planes

$$U=\operatorname{Spec} k[x,y], \qquad U_1=\operatorname{Spec} k[x_1=1/x,\ y_1=y],$$
$$U_2=\operatorname{Spec} k[x_2=xy^n,\ y_2=1/y], \qquad W=\operatorname{Spec} k[u=1/y,\ v=1/x_2].$$

We denote by P_1, P_2, P the origins of the latter three planes, respectively. The fibres F of $\mathbb{F}_n\to\mathbb{P}^1$ are given by $y=a$, $a\in\mathbb{P}^1$; we denote by F_∞ the fibre at infinity of U. The section given by $x=0$ on U we denote by M; in case $n>0$, M is the unique irreducible curve on $\mathbb{F}_n$ with self-intersection $-n$. The section at infinity of U we denote by M_∞.

For any $N\geq 0$, we have the diagram

$$\begin{array}{ccccc} \mathbb{F}_N\cong\theta^{-1}\mathbb{F}_N & \xrightarrow{\pi'_N} & \mathbb{F}_{Np} & \xrightarrow{\pi_N} & \mathbb{F}_N \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{P}^1 & \xrightarrow[\theta']{} & \mathbb{P}^1 \end{array}$$

where $\pi_N\circ\pi'_N$ and θ' are Frobenius morphisms, and the square is a pullback.

If we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\cong} & Y \\ \varphi\downarrow & & \downarrow\varphi' \\ X & \xrightarrow[\cong]{} & X \end{array}$$

of morphisms of varieties, then we say φ and φ' are *equivalent*.

2.2. Theorem. *Let $\varphi: Y \to \mathbb{F}_n$ be an inseparable degree p covering, with Y nonsingular.*

(i) *If $p \nmid n$, then φ is equivalent to π_n.*

(ii) *If $p \mid n$, then φ is equivalent to either π_n or $\pi'_{n/p}$.*

Proof. We search for derivations D without isolated singularities on $\mathbb{F}_n$. Let $D = f(\partial/\partial x) + g(\partial/\partial y)$, where $f, g \in k[x, y]$ do not meet in U. If $g = 0$, $D \sim \partial/\partial x$. One checks that D has no isolated singularity on $\mathbb{F}_n$, and that the corresponding quotient map is $\pi'_n: \mathbb{F}_n \to Y$. If $f = 0$, $D \sim \partial/\partial y$ has isolated singularities (at P, P_2) $\Leftrightarrow$ $p \nmid n$. If $p \mid n$, the corresponding quotient map is $\pi_{n/p}: \mathbb{F}_n \to Y$. So in these cases, 2.2 follows from 1.3.

Henceforth we assume f and g are nonzero.

Given a nonzero element of $k[x, y]$, we will denote by the same letter the given element and the effective divisor on $\mathbb{F}_n$ defined by it. We denote linear equivalence by $\sim$.

2.2.1. Let $r = \deg_x f$, $s = \deg_x g$. Let

$$f = \sum f_{ij} x^i y^j, \qquad g = \sum g_{ij} x^i y^j,$$

$$m = \min\{ni - j \mid f_{ij} \neq 0\}, \qquad l = \min\{ni - j \mid g_{ij} \neq 0\}.$$

We have $f \sim rM + (nr - m)F$ and $g \sim sM + (ns - l)F$, so

1. $(f \cdot g)_{\mathbb{F}_n} = nrs - (rl + sm)$. Note that
2. $l \leq ns$, $m \leq nr$, and
3. $l \leq 0$ or $m \leq 0$,

the last statement following from the fact that x is not a common factor of f and g.

2.2.2. Local equations f_i, g_i for f and g on U_i, and $\hat{f}, \hat{g}$ on W are given by

$$f = f_1 x_1^{-r} = f_2 y_2^m = \hat{f} u^m v^{-r}$$

and

$$g = g_1 x_1^{-s} = g_2 y_2^l = \hat{g} u^l v^{-s}.$$

In several computations in the case $p \nmid n$ the polynomial $h := fy + ngx$ will crop up. We record some data about h.

$h = \sum h_{ij} x^i y^j \neq 0$. (Otherwise, f, g meet at the origin of U.) Put $d = \deg_x h$, $e = \min\{ni - j \mid h_{ij} \neq 0\}$, $h = h_1 x_1^{-d} = h_2 y_2^e = \hat{h} u^e v^{-d}$. Note that $e \geq \min\{l + n, m - 1\}$. By 2.2.2 we have:

2.2.3.

$$\begin{aligned} h &= f_1 x_1^{-r} y_1 + n g_1 x_1^{-s-1} \\ &= f_2 y_2^{m-1} + n g_2 x_2 y_2^{l+n} = \hat{f} u^{m-1} v^{-r} + n\hat{g} u^{l+n} v^{-s-1}. \end{aligned}$$

We rewrite D in each coordinate system.

2.2.4.

1. $$D = -f_1 x_1^{2-r} \frac{\partial}{\partial x_1} + g_1 x_1^{-s} \frac{\partial}{\partial y_1}.$$

2. $$D = (f_2 y_2^{m-n} + n g_2 x_2 y_2^{l+1}) \frac{\partial}{\partial x_2} - g_2 y_2^{l+2} \frac{\partial}{\partial y_2}.$$

3. $$-D = \hat{g} u^{l+2} v^{-s} \frac{\partial}{\partial u} + (\hat{f} u^{m-n} v^{2-r} + n \hat{g} u^{l+1} v^{1-s}) \frac{\partial}{\partial v}.$$

The conditions that D have no isolated singularity on the line $L_1 = M_\infty \setminus \{P\}$ become:

2.2.5. 1. $r > s + 2$ and $f \cap L_1 = \emptyset$, or
2. $r < s + 2$ and $g \cap L_1 = \emptyset$, or
3. $r = s + 2$ and $f \cap g \cap {}_1 = \emptyset$.

The case $p \nmid n$. The conditions that D have no isolated singularity on the line $L_2 = F_\infty \setminus \{P\}$ become:

2.2.6. 1. $m - n < l + 1$ and $f \cap L_2 = \emptyset$, or
2. $m - n = l + 1$ and
   (a) $e = m - 1$ and $h \cap L_2 = \emptyset$, or
   (b) $e = m$ and $g \cap h \cap L_2 = \emptyset$, or
   (c) $e > m$ and $g \cap L_2 = \emptyset$.

Proof of 2.2.6: If $m - n > l + 1$, then by 2.2.4.2, D has a singularity at P_2. 2.2.6.1 is clear. If $m - n = l + 1$, then by 2.2.3 and 2.2.4.2,

$$D \sim h_2 y_2^{e-(m-1)} \frac{\partial}{\partial x_2} - g_2 y_2 \frac{\partial}{\partial y_2};$$

2.2.6.2 follows.

We proceed to eliminate all possibilities for D.

If $m-n<l+1$ and $r<s+2$, one sees at once from 2.2.4.3 that D is singular at P. If instead $r\geq s+2$, then by 2.2.6.1, $f\cap L_2=\emptyset$, and 2.2.4.3 requires that $\hat{f}(P)\neq 0$. Then $r=f\cdot F_\infty=0$ and $s<0$, which cannot be.

2.2.7. We are reduced to the case $m-n=l+1$. Note that by 2.2.1.3, $l\leq 0$.

2.2.7.1. Suppose $s+2\geq r$. Then

$$D\sim \hat{g}u\frac{\partial}{\partial u}+(\hat{f}v^{s+2-r}+n\hat{g}v)\frac{\partial}{\partial v}=\hat{g}u\frac{\partial}{\partial u}+\hat{h}u^{e-(m-1)}v^{s+2-d}\frac{\partial}{\partial v},$$

by 2.2.3. If $e=m-1$, D is singular at P. Suppose $e\geq m$. Then we must have $\hat{g}(P)\neq 0$ since $s+2>d$. With 2.2.5.2 this gives $0=g\cdot M_\infty=(sM+(ns-l)F)\cdot(M+nF)=ns-l$. So $ns=l\leq 0$. Hence $s=l=0$, since $p\nmid n$. Then $r\leq 1$, and by 2.2.1.2, $n+1=m\leq nr\leq n$, an absurdity.

2.2.7.2. Suppose $r=s+2$. Then

$$D\sim \hat{g}u\frac{\partial}{\partial u}+\hat{h}u^{e-(m-1)}\frac{\partial}{\partial v}$$

since $d=r$. If $e=m-1$, we must have $\hat{h}(P)\neq 0$; with 2.2.6.2(a) this gives $0=h\cdot F_\infty=d$, and $s<0$. If $e>m$, we must have $\hat{g}(P)\neq 0$. Then by 2.2.6.2(c), $s=g\cdot F_\infty=0$. Also $0=(f\cdot g)_{\bar{F}_n}=-rl$ by 2.2.5.3 and 2.2.1.1. So $l=0$. But then $g\in k^*$, no terms of fy and ngx can cancel, and $e=m-1$, a contradiction. Hence we may assume $e=m$. $\hat{g}$ and $\hat{h}$ must not meet at P. By 2.2.6.2(b), g and h do not meet on F_∞, and $(f\cdot g)_{L_1}=0$ by 2.2.5.3. From 2.2.3 we have $h_2y_2=f_2+ng_2x_2$ and $\hat{h}u=\hat{f}+n\hat{g}v$. Hence

$$(f\cdot g)_{L_2}=(g_2\cdot y_2)_{L_2}=\deg_{x_2}\left(\sum_{ni-j=l}g_{ij}x_2^i\right),$$

and

$$(\hat{f}\cdot\hat{g})_P=(\hat{g}\cdot u)_P=\operatorname{ord}_v\left(\sum_{ni-j=l}g_{ij}v^{s-i}\right).$$

Summing, we have $s=(f\cdot g)_{F_\infty}=nrs-(rl+sm)$ by 2.2.1.1. Expressing r,m in terms of s,l, we have

$$(s+2)(ns-l)=s(l+n+2), \tag{$\blacklozenge$}$$

from which it follows that $ns-l<l+n+2$, $n(s-1)<2l+2\leq 2$, and $n(s-1)\leq 1$. $n>0$ since $p\nmid n$, so $s=0,1$, or 2. $s=2$ gives $n=1$; then ($\blacklozenge$) gives $6l=2$, which is nonsense. $s=0$ and ($\blacklozenge$) imply $l=0$. Then $g\in k^*$; as in the last paragraph, this is impossible. $s=1$ and ($\blacklozenge$) imply $n=1$ and $l=0$. Hence $m=2$, and $x^2\mid f$. So g has nonzero constant term and h nonzero 'x' term. So $2=e\leq n\cdot 1-0=1$.

2.2.7.3. Suppose $r>s+2$. Then

$$D \sim \hat{g}uv^{r-s-2}\frac{\partial}{\partial u}+\hat{h}u^{e-(m-1)}\frac{\partial}{\partial v}$$

by 2.2.4.3, 2.2.3, and the fact that $d=r$. If $e=m-1$, the argument of 2.2.7.2 works. If $e>m$, D is singular at P. Hence we may assume $e=m$; the details of this case are as in 2.2.7.2, and are left to the reader.

The case $p\nmid n$. In view of 2.2.4.2, the conditions that D have no isolated singularity on L_2 become:

2.2.8. 1. $m-n>l+2$ and $g\cap L_2=\emptyset$, or
2. $m-n<l+2$ and $f\cap L_2=\emptyset$, or
3. $m-n=l+2$ and $f\cap g\cap L_2=\emptyset$.

By 2.2.4.3 we have

$$D \sim \hat{g}u^{l+2}v^{-s}\frac{\partial}{\partial u}+\hat{f}u^{m-n}v^{2-r}\frac{\partial}{\partial v}\,.$$

Thus if $l+2>m-n$ and $r<s+2$, or $l+2<m-n$ and $r>s+2$, then D has an isolated singularity at P. If $l+2>m-n$ and $r\geq s+2$, we must have $\hat{f}(P)\neq 0$, so by 2.2.8.2, $0=f\cdot F_\infty=r$, and $s<0$.

If $l+2\leq m-n$ and $r<s+2$, we must have $\hat{g}(P)\neq 0$. By 2.2.5.2, $0=g\cdot M_\infty=ns-l$. By 2.2.1.2, $n(s+1)\geq nr\geq m\geq l+n+2=n(s+1)+2$.

If $l+2=m-n$ and $r>s+2$, we must have $\hat{f}(P)\neq 0$, so by 2.2.5.1, $0=f\cdot M_\infty=nr-m$. By 2.2.8.3 and 2.2.1.1, $0=(f\cdot g)_{\mathbb{F}_n}=nrs-sm-rl=-rl$. So $l=0$, and $nr=m=n+2$, hence $n(r-1)=2$. But $r\geq 3$, so $n=1\not\equiv 0 \pmod p$.

Finally we have the case $r=s+2$, $l+2\leq m-n$. We have:

2.2.9. $(s+2)(ns-l)-s(l+n+2)=r(ns-l)-sm=(f\cdot g)_{\mathbb{F}_n}=0$. (If $l+2=m-n$, this follows from 2.2.5.3, 2.2.8.3, and the requirement that D have no isolated singularity at P. If $l+2<m-n$, it follows from 2.2.5.3, 2.2.8.1, the requirement at P, and the fact that $0=g\cdot F_\infty=s$.) One deduces that $ns-l<l+n+2$, hence $n(s-1)<2l+2\leq 2$.

If $n=0$, then by 2.2.9, $l(s+2)+s(l+2)=0$, so $(l+1)(s+1)=1$, and $l=0$. By 2.2.1.2, $2=l+n+2\leq m\leq nr=0$.

So we may asume $n\geq 2$ and $s=0$ or 1. s cannot be 1, for then $m\leq nr=rl+m\leq m$, so $l=0$ and $3n=n+2$ (by 2.2.9), whence $n=1$. Therefore $s=0$, $r=2$, $l=0$ by 2.2.9, and $m\geq n+2$. Looking to the Newton diagrams of f and g, we have $g\in k^*$ and $f=\varphi(y)x^2$, with $\deg\varphi\leq n-2$. We conclude that

$$D \sim \frac{\partial}{\partial y}+\varphi(y)x^2\frac{\partial}{\partial x}=\frac{\partial}{\partial y_1}-\varphi(y_1)\frac{\partial}{\partial x_1}\,,$$

for some φ of degree $\le n-2$. Now $D^p(y_1)=0$ and $D^p(x_1)=-d^{p-1}\varphi/dy_1^{p-1}$, hence

$$D \text{ is } p\text{-closed} \Leftrightarrow \varphi \text{ is a derivative.}$$

Suppose this is so, and let H be an antiderivative of φ. Putting

$$\tilde{y}=y_1, \qquad \tilde{x}=x_1+H(y_1), \tag{►}$$

we have $D\sim\partial/\partial\tilde{y}$. If we choose H of degree $\le n$ (as we may, since $\deg\varphi\le n-2$), then (►) defines an automorphism of $\mathbb{F}_n$ ([2], (4.4.2), p. 65). This completes the proof of Theorem 2.2.

Our results do not seem to extend readily to arbitrary rational surfaces. The example $\mathbb{F}_1\to\mathbb{P}^2$ and the following one point to a non-trivial relationship between the sets of derivations without isolated singularities on each of two surfaces which differ by a single blowing-up.

2.3. Example. Let $Z\to\mathbb{F}_0=\mathbb{P}_k^1\times\mathbb{P}_k^1$ be the blowing-up of a point. Then any nonzero derivation D on Z has isolated singularities.

We sketch the proof. By homogeneity of $\mathbb{F}_0$, we may assume that the blown-up point is the point P in the proof of 2.2, and may take the u, v of that proof as local parameters at P. One can write down precisely the conditions that $D\sim\alpha(\partial/\partial u)+\beta(\partial/\partial v)$ (α,β coprime) have no isolated singularity on the exceptional fibre $E\subset Z$. Comparison with the possibilities for D gotten by combining 2.2.5 and 2.2.8 gives 2.3

3. The case $Z=C\times\mathbb{P}^1$, C non-rational

Throughout this section, C will stand for a complete nonsingular curve of genus $g\ge 1$. We begin by collecting a few facts about k-derivations of $k(C)$.

3.1. We call a derivation $\delta\ne 0$ of $k(C)$ *normalized* if $-(\delta)$ is an effective divisor. Note that then $-(\delta)$ is an effective canonical divisor on C, and that such divisors, and hence normalized derivations, exist on C.

3.2. For each $q\in C$ we fix a parameter ξ_q. A polynomial

$$P_q=a_s\xi_q^{-s}+a_{s-1}\xi_q^{-s+1}+\cdots+a_1\xi_q^{-1},$$

$s>0$, $a_i\in k$, is then called *a principal part at q*. If $f\in k(C)$, we can write

$$f=P_q+g,$$

where P_q is a principal part and g is regular at q. We then call $P_q=P_q(f)$ the *principal part of f at q*. We denote by

$$\mathcal{P}=\{(P_q)_{q\in C}\mid P_q=0 \text{ for almost all } q\}$$

the vector space of all principal parts on C and by

$$\mathscr{P}_0 = \{(P_q(f))_{q\in C} \mid f \in k(C)\}$$

the subspace of principal parts of rational functions.

3.3. We call a derivation $\delta \neq 0$ on a field L (of characteristic p) *additive* if $\delta^p = 0$ and *multiplicative if* $\delta^p = \delta$. We have

3.3.1. Lemma. *Let δ be p-closed.*

(i) *δ is additive if and only if there exists $f \in L$ with $\delta(f) = 1$.*

(ii) *δ is multiplicative if and only if there exists $f \in L^*$ with $\delta(f) = f$.*

Proof. The 'if' parts of both assertions are obvious. We prove the 'only if' parts.

(i) Suppose $\delta^p = 0$. There exists $f \in L$ such that $\delta(f) \neq 0$. Hence there exists i, $1 \leq i < p$, such that $\delta^i(f) \neq 0$ and $\delta^{i+1}(f) = 0$. Then $\delta(\delta^{i-1}(f)/\delta^i(f)) = 1$.

(ii) Suppose $\delta^p = \delta$. There exists $g' \in L$ such that $\delta(g') \neq 0$. Then $\delta^i(g') \neq 0$ for all i. Put $g = \delta^{p-1}(g')$. Then $\delta^{p-1}(g) = g$ and hence the operator δ on the (finite-dimensional) L^δ-vector space spanned by $\{\delta^i(g) \mid i \geq 0\}$ has an eigenvalue ω with $\omega^{p-1} = 1$. Let $f' \in L^*$ be a corresponding eigenvector. Then $f := (f')^{1/\omega}$ makes sense, and $\delta(f) = f$.

3.4. Lemma. *Let δ be a nonzero derivation of $k(C)$. Let $f \in k(C)$ such that $\delta(f) \neq 0$.*

(i) *$(\delta(f)/f) - (\delta)$ is an effective (canonical) divisor if and only if there exists a divisor E on C such that $(f) = pE$.*

(ii) *$(\delta(f)) - (\delta)$ is an effective (canonical) divisor if and only if all principal parts of f are p-th powers.*

Proof. Let $q \in C$ and let ξ be a local parameter at q. We write

$$f = \xi^r u,$$

where u is a unit at q, and

$$\delta = \varepsilon \frac{\partial}{\partial \xi}.$$

So ε is a local equation for (δ) at q.

(i) We have

$$\delta(f)/f = \varepsilon\left(r/\xi + u^{-1}\frac{\partial u}{\partial \xi}\right)$$

with $u^{-1}(\partial u/\partial \xi) \in \mathcal{O}_{C,q}$. Hence $\varepsilon^{-1}\delta(f)/f$ has no pole at q if and only if $r \equiv 0 \pmod p$.

(ii) We have

$$\varepsilon^{-1}\delta(f) = ru\xi^{r-1} + \xi^r \frac{\partial u}{\partial \xi}$$

with $(\partial u/\partial \xi) \in \hat{\mathcal{O}}_{C,q}$. If $r \geq 0$, there is nothing to prove. So assume $r<0$ and write

$$u = \sum_{i \geq 0} u_i \xi^i, \quad u_0 \neq 0,$$

in $\hat{\mathcal{O}}_{C,q}$. Then

$$P := \xi^r \sum_{0 \leq i < -r} u_i \xi^i$$

is the principal part of f at q. Clearly $\varepsilon^{-1}\delta(f)$ has no pole at q if and only if $r \equiv 0 \pmod p$ and $u_i = 0$ for $0 < i < -r$, $i \not\equiv 0 \pmod p$.

3.5. By combining 3.3.1 and 3.4 we find the following results concerning the existence of normalized additive or multiplicative derivations on C. Here δ is a fixed nonzero derivation on C.

(i) If $\delta(f) \neq 0$ and $(\delta(f)/f) - (\delta)$ is effective, then $\delta' = (f/\delta(f)\delta$ is a normalized multiplicative derivation. Since divisors E on C such that pE is linearly equivalent to 0 are given by points of order p on the Jacobian of C, such points provide normalized multiplicative derivations on C.

(ii) If $\delta(f) \neq 0$ and $(\delta(f)) - (\delta)$ is effective, then $\delta' = \delta/\delta(f)$ is a normalized additive derivation. It is well known that there is an isomorphism (see [4], p. 27)

$$\mathcal{P}/\mathcal{P}_0 \to H^1(C, \mathcal{O}_C).$$

(Given $P \in \mathcal{P}$, choose an open covering $\{U_i\}$ of C such that there exist rational functions f_i on U_i with principal parts equal to those of P. The image of P in $H^1(C, \mathcal{O}_C)$ is the class of the cocycle $f_i - f_j$.) Now $H^1(C, \mathcal{O}_C)$ has a natural p-th power map whose kernel corresponds to the 'supersingular' part of the Jacobian of C (the deficiency in points of order p) and that is given on $\mathcal{P}$ by $(P_q)_{q \in C} \mapsto (P_q^p)_{q \in C}$. Hence the supersingular part of the Jacobian of C provides normalized additive derivations on C.

3.6. Let $\pi : Z \to Y$ be an inseparable degree p covering of nonsingular surfaces, $q \in Z$, $q' = \pi(q)$, $\vartheta = \mathcal{O}_{Z,q}$ and $\vartheta' = \mathcal{O}_{Y,q'}$. By the discussion in Section 1 there exists a regular system of parameters (ξ, η) for ϑ such that (ξ^p, η) is a regular system of parameters for ϑ'. Let $\varphi' : Y' \to Y$ be the blowing up of q' and $\varphi : Z' \to Z$ the blowing up of p points infinitely near to q along the curve $\eta = 0$. (Note that this is well defined. Another parameter with the same properties as η has contact at least p with η at q.) Let G be the exceptional curve of φ' and $H_1, \ldots, H_p$ the exceptional curves of φ, labelled in their order of appearance. We then have the following result, the proof of which we leave to the reader (see also [5], Lemma 2.4).

Lemma. *π induces a morphism $\Pi : Z' \to Y'$. Π induces an isomorphism $H_p \to G$ and maps $H_{p-1}, \ldots, H_1$ to the point on G corresponding to the direction $\xi^p = 0$.*

3.7. Let $Z = C \times \mathbb{P}^1$ and write $k(\mathbb{P}^1) = k(t)$. Let δ be a nonzero derivation on $k(C)$. We extend δ and $\partial/\partial t$ trivially to derivations on $k(Z) = k(C)(t)$, which we denote by the same letters. Any derivation on Z is then equivalent to $\partial/\partial t$ or to a derivation

$$D = \delta + h \frac{\partial}{\partial t}$$

with $h \in k(Z)$. We put

$$K = -(\delta) \times \mathbb{P}^1 \quad \text{and} \quad C_\infty = C \times \{\infty\},$$

where $\infty \in \mathbb{P}^1$ is given by $1/t = 0$.

3.7.1. Lemma. *Let $D = \delta + h(\partial/\partial t)$, with δ normalized.*

(i) *If $g = 1$, then D has no isolated singularity on Z if and only if $h \in k(t)$.*

(ii) *If $g > 1$, then D has no isolated singularity on Z if and only if $h = 0$ or $(h) + K + 2C_\infty \geq 0$.*

Proof. It is clear that δ has no isolated singularity on Z. We may assume, therefore, that $h \neq 0$.

In this proof we call 'vertical' a curve on Z of the form $\{q\} \times \mathbb{P}^1$, $q \in C$, and 'horizontal' a curve of the form $C \times \{q\}$, $q \in \mathbb{P}^1$. Write $(h) = E_0 - E_\infty$, where E_0 and E_∞ are effective divisors without common component. Let $F = \inf\{E_\infty, K + 2C_\infty\}$. Since $h(\partial/\partial t) = E_0 - E_\infty + 2C_\infty$ we have at a point $q \in Z$ with local parameters ξ (along C) and η (along $\mathbb{P}^1$)

$$D = b_q\left(f_1 \frac{\partial}{\partial \xi} + f_2 \frac{\partial}{\partial \eta}\right),$$

where b_q is a local equation for $-K - E_\infty + F$, f_1 is a local equation for $E_1 := E_\infty - F$, f_2 is a local equation for $E_2 := E_0 + 2C_\infty + K - F$ and, by construction, $\mathrm{GCD}(f_1, f_2) = 1$.

Case 1: The components of E_0 are neither all horizontal nor all vertical. Then D has no isolated singularity on Z if and only if $E_1 = 0$, i.e. if and only if $E_\infty \leq K + 2C_\infty$.

Case 2: All components of E_0 are vertical. Then $h \in k(C)$, E_∞ is vertical, hence F is vertical, and C_∞ is a horizontal component of E_2. Hence again D has no isolated singularity if and only if $E_1 = 0$, or $E_\infty \leq K$.

Case 3: All components of E_0 are horizontal. Then $h \in k(t)$, E_∞ is horizontal, hence F is horizontal. If $E_1 = 0$, we are done as in cases 1 and 2. If $E_1 \neq 0$, i.e. if $(h)_\infty \not\leq 2C_\infty$, then D has no isolated singularity if and only if $K = 0$, i.e. $g = 1$.

The above proof gives the following corollary:

3.7.2. Corollary. *Suppose $D = \delta + h(\delta/\delta t)$ has no isolated singularity on Z. Then $(D) = -K$ if $g > 1$ and $(D) = -(h)_\infty + \inf\{(h)_\infty, 2C_\infty\}$ if $g = 1$.*

3.8. We keep the notation of 3.7. Suppose now $D = \delta + h(\partial/\partial t)$ has no isolated singularity on Z. It does not seem to be easy in general to determine when D is p-closed, and we will make the simplifying assumption

$$h = h_1 h_2$$

with $h_1 \in k(t)$ and $0 \neq h_2 \in k(C)$. (By 3.7.1 this is no restriction if $g=1$.) Then $(h_2)_\infty \leq K$ and hence $h_2^{-1}\delta$ is a normalized derivation on C. So we are free to replace D by $h_2^{-1}D$, i.e. we may assume $h_2 = 1$ and $h = h_1 \in k(t)$. If $g > 1$, we then have $(h)_\infty \leq 2C_\infty$ by 3.7.1. We therefore first consider

3.8.1. *The case* $h \in k[t]$, $\deg_t h \leq 2$. By an appropriate choice of t we can reduce our discussion to the following two cases:

(a) $$\Delta := h\frac{\partial}{\partial t} = \frac{\partial}{\partial t} \qquad \text{(additive case)},$$

(m) $$\Delta := h\frac{\partial}{\partial t} = t\frac{\partial}{\partial t} \qquad \text{(multiplicative case)}.$$

Since $D = \delta + \Delta$ with $\delta\Delta = \Delta\delta$, we have $D^p = \delta^p + \Delta^p$ and D is p-closed if and only if $\delta^p = 0$ in case (a) or $\delta^p = \delta$ in case (m).

Case (m). Let E be a divisor on C and write $E = E_0 - E_\infty$ where $E_0 = \sum r_i q_i$ and $E_\infty = \sum r_i' q_i'$ are effective divisors without common component. The following is a standard construction in the theory of ruled surfaces (see for instance [4], III §1). For $q_i \in \operatorname{Supp} E_0$ (resp. $q_i' \in \operatorname{Supp} E_\infty$) we blow up r_i points infinitely near to $(q_i, 0)$ on $C \times \{0\}$ (resp. r_i' infinitely near to (q_i', ∞) on $C \times \{\infty\}$). Let $E_{i,1}, \ldots, E_{i,r_i}$ (resp. $E_{i,1}', \ldots, E_{i,r_i'}'$) be the exceptional curves. We then shrink successively the proper transform of $\{q_i\} \times \mathbb{P}^1$ (resp. $\{q_i'\} \times \mathbb{P}^1$) and $E_{i,1}, \ldots, E_{i,r_i-1}$ (resp. $E_{i,1}', \ldots, E_{i,r_i'-1}'$). There results a ruled surface which we call Z_E.

Suppose now that $E = pE'$, where E' is a divisor on C. Clearly this is equivalent to: There exists a divisor E_θ on θC (the Frobenius of C) such that $E = \theta^*(E_\theta)$. We can now apply the above construction to $Y = \theta C \times \mathbb{P}^1$ and E_θ and find by repeated application of 3.6:

There exists a commutative diagram of morphisms

$$\begin{array}{ccc} Z_E & \xrightarrow{\ \Pi\ } & Y_{E_\theta} \\ \downarrow & & \downarrow \\ C & \longrightarrow & \theta C \end{array}$$

with Π *an inseparable covering of degree* p.

It is well known that $Z_E \cong C \times \mathbb{P}^1$ if and only if E is linearly equivalent to 0. Suppose then $E = \theta^*(E_\theta)$ and $E = (f)$, $f \notin k(C)^p$. It is easy to see that then $Y_{E_\theta} \cong Z^D$, where $D = \delta + t(\partial/\partial t)$ with δ normalized and $\delta(f) = f$. (We have $D(t/f) = 0$, and the birational map $Z \to Z = Z_E$ given by $(q, t) \mapsto (q, t/f(q))$ factors into quadratic transformations precisely as described in the construction of Z_E.)

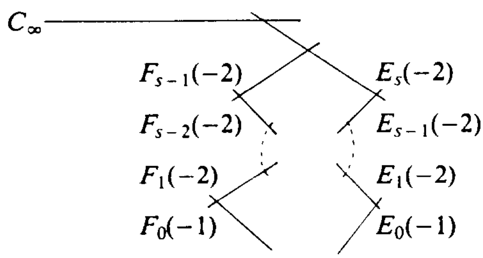
Case (a). Let $P=(P_q)_{q\in C}$ be a principal part on C. Suppose $P_q\neq 0$. Let ξ be a local parameter for C at q and η a local parameter for $\mathbb{P}^1$ at ∞. Write

$$P_q=a_s\xi^{-s}+\cdots+a_1\xi^{-1},\quad a_s\neq 0,$$

and blow up $2s$ points on $C\times\mathbb{P}^1$ infinitely near to (q,∞) along the branch

$$\xi^s=\eta(a_s+a_{s-1}\xi+\cdots+a_1\xi^{s-1}).$$

The following diagram illustrates this situation. (Numbers in parentheses give self-intersections.)



Here $E_0=\{q\}\times\mathbb{P}^1$ and the exceptional curves (in order of appearance) are

$$E_1,\ldots,E_{s-1},E_s,F_{s-1},\ldots,F_0. \qquad (\blacktriangle)$$

We can now shrink successively E_0 and the curves of ($\blacktriangle$) with the exception of F_0. Doing this for each $q\in\operatorname{Supp}P$, we obtain a ruled surface which we denote Z_P.

Suppose now that $P=P'^p$, where P' is a principal part on C. Clearly this is so if and only if P is the pullback via θ of a principal part P_θ on θC. (To fix the ideas we choose local parameters on θC that are p-th powers of local parameters chosen at corresponding points of C.) We now apply the above construction to $Y=\theta C\times\mathbb{P}^1$ and find by repeated application of 3.6:

There exists a commutative diagram of morphisms

$$\begin{array}{ccc} Z_P & \xrightarrow{\ \Pi\ } & Y_{P_\theta} \\ \downarrow & & \downarrow \\ C & \longrightarrow & \theta C \end{array}$$

with Π an inseparable covering of degree p.

It is not hard to check that $Z_P\cong C\times\mathbb{P}^1$ if and only if $P\in\mathscr{P}_0$, i.e. if and only if P is the principal part of a rational function $f\in k(C)$. If this is the case we have $Y_{P_\theta}\cong Z^D$, where $D=\delta+(\partial/\partial t)$ with δ normalized and $\delta(f)=1$. (We have $D(t-f)=0$, and the birational map $Z\to Z_P=Z$ given by $(q,t)\mapsto(q,t-f(q))$ factors into quadratic transformations precisely as described in the construction of Z_P.)

3.8.2. *The case $g=1$.* If δ is a normalized derivation on C, we have $(\delta)=0$ and hence $\delta^p=a\delta$ with $a\in k$. We may assume, therefore, that either

(m) $\delta^p=\delta$ (C not supersingular)

or

(a) $\delta^p=0$ (C supersingular).

Suppose $D=\delta+h(\partial/\partial t)$ is without isolated singularities on Z. Then $h\in k(t)$ by Lemma 3.7.1, and D is p-closed if and only if $\Delta^p=\Delta$ in case (m) or $\Delta^p=0$ in case (a), where $\Delta=h(\partial/\partial t)$. By a formula of Hochschild (see [3], p. 191) this is equivalent to $\Delta^{p-1}(h)=h$ in case (m) and $\Delta^{p-1}(h)=0$ in case (a). Possible (though if $p>2$ not all) solutions are $h=t+g^p$ and $h=g^p$ respectively, $g\in k(t)$.

Now suppose D is p-closed and let $\pi:Z\to Y=Z^D$ be the quotient morphism. Let L be a canonical divisor on Y. By [6], Corollary 1 on p. 1213, we have

$$\pi^*(L)\sim -2C_\infty-(p-1)(D).$$

Applying Corollary 3.7.2, we find that

$$\pi^*(L)\sim C\times Q,$$

where Q is a divisor on $\mathbb{P}^1$ of degree

$$d=-2+(p-1)d_1$$

with

$$d_1=\deg((h)_\infty-\inf\{(h)_\infty,2\infty\})\geq 0.$$

Now for $n\geq 0$,

$$h^0(Y,\mathcal{O}(nL))\leq h^0(Z,\mathcal{O}(\pi^*(nL)))\leq h^0(Y,\mathcal{O}(pnL))$$

(the middle term is just $h^0(\mathbb{P}^1,\mathcal{O}(nQ))$) and hence $\kappa(Y)$, the Kodaira dimension of Y, is 1 if $d>0$, whereas $\kappa(Y)=0$ if $d=0$.

If $d=0$ we have one of the following:

(i) $p=2$ and $d_1=2$. The possibilities for h are (with appropriate choice of t) in case (m): $h=t+g^2$, $g\in k[k]$, $\deg g=2$ and in case (a): $h=g^2$, $g\in k[t]$, $\deg g=2$ or $h=g^2/t^2$, $t\nmid g\in k[t]$, $\deg g\leq 2$.

(ii) $p=3$ and $d_1=1$. The possibilites for h are (with appropriate choice of t) in case (m): $h=t+g^3$, $g\in k[t]$, $\deg g=1$ or $h=t+c/t$, $c\in k^*$; and in case (a): $h=g^3$, $g\in k[t]$, $\deg g=1$ or $h=c/t$, $c\in k^*$.

One checks easily that there are no D with $d=-1$ (and hence $p=2$, $d_1=1$). The case $d=-2$, or $d_1=0$, has been treated in 3.8.1.

4. Subbundles of the tangent bundle

Let Z be a nonsingular surface. Derivations of $k(Z)$ are naturally identified with rational sections of the tangent bundle T_Z of Z, i.e. with 'rational' homomorphisms

$$\mathcal{O}_Z \to T_Z.$$

More precisely, if $\mathscr{L}$ is a line bundle on Z, there is a natural one-to-one correspondence between homomorphisms with only isolated zeros

$$\alpha' : \mathcal{O}_Z \to T_Z \otimes \mathscr{L}^{-1}$$

or equivalently, homomorphisms with only isolated zeros

$$\alpha : \mathscr{L} \to T_Z$$

on the one hand and equivalence classes of derivations D (see Section 1) with $\mathscr{L} \cong \mathcal{O}(D)$) on the other. (One identifies homomorphisms which differ by an automorphism of $\mathscr{L}$.) We call α, or, if there is no danger of confusion, the line bundle $\mathscr{L}$, *integrable* if the corresponding derivation is p-closed (see [7], §3). It is clear that D has no isolated singularity on Z if and only if the corresponding α embeds $\mathscr{L}$ as a sublinebundle in T_Z, i.e. if and only if $T_Z/\alpha(\mathscr{L})$ is again a line bundle. We can therefore interpret the results of Section 1 as saying that inseparable coverings $\pi : Z \to Y$ of degree p, with Y nonsingular, are given by integrable subbundles of the tangent bundle of Z.

We can therefore reformulate the result of the proof of Theorem 2.1 as follows:

4.1. Theorem. *$T_{\mathbb{P}^2}$ has no sublinebundles.*

If $Z \to C$ is a ruled surface over the nonsingular curve C, there is a well-known exact sequence

$$0 \longrightarrow \Phi \xrightarrow{\ \alpha\ } T_Z \xrightarrow{\ \beta\ } N \longrightarrow 0 \qquad (\dagger)$$

where N is the pullback of the tangent bundle of C and Φ is the line bundle of tangents along the fibres of Z. It is easy to see that α is integrable. (It corresponds to $D = \partial/\partial t$, with $k(Z) = k(C)(t)$.)

The results of Section 2 on rational ruled surfaces now translate as follows.

4.2. Theorem. (i) *If $p \nmid n$, then the sublinebundles of $T_{\mathbb{F}_n}$ are precisely those derived from $\alpha : \Phi \to T_{\mathbb{F}_n}$ by an automorphism of $\mathbb{F}_n$. They are all integrable.*

(ii) *If $p \mid n$, then the sequence ($\dagger$) splits and there exists an integrable section $\sigma : N \to T_{\mathbb{F}_n}$. The integrable sublinebundles of $T_{\mathbb{F}_n}$ are those derived from α or σ by an automorphism of $\mathbb{F}_n$.*

It is easy to see that if a ruled surface $Z \to C$ is the pullback by the Frobenius of C of a ruled surface $Y \to \theta C$, then ($\dagger$) splits. Our investigations suggest:

4.3. Conjecture. The converse is true.

4.4. A. Lascu has shown us how to use the calculus of characteristic classes to derive necessary conditions on a line bundle $\mathscr{L}$ to appear as sublinebundle of T_Z: The reverse Chern polynomial of Z vanishes when evaluated on $\mathscr{L}$. One obtains a particularly concise proof of Theorem 4.1 in this way: The reverse Chern polynomial of $\mathbb{P}^2$ is $t^2+3lt+3$, where l is the class of a line. Evaluation on a line bundle of degree n leads to $n^2+3n+3=0$, which is impossible.

Corollary 3.7.2, which classifies the possible sublinebundles of T_Z when $Z=C\times\mathbb{P}^1$, C a curve of positive genus, could also be proved by this technique. It seems to us, however, that most of our detailed results, particularly those concerned with integrability conditions, require a more direct method of attack.

Acknowledgement

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Notes added in proof

(1) Theorem 4.1 is not new, only the use made of it in 2.1. It appears as an exercise, for instance, in R. Hartshorne, Algebraic Geometry (Springer, Berlin–New York, 1977).

(2) Conjecture 4.3 is not true unless one asks for integrable sections of (†). In fact, (†) splits for a 'general' ruled surface (of even degree if char $k=2$). This holds even if char $k=0$.

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